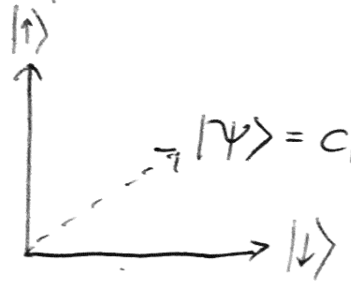
SPIN-½ DEGREES OF FREEDOM

Recap – QM wavefunctions (wavevectors) live in 'Hilbert space', a complex vector space.

Spin ½ ⟹ Two dimensional space

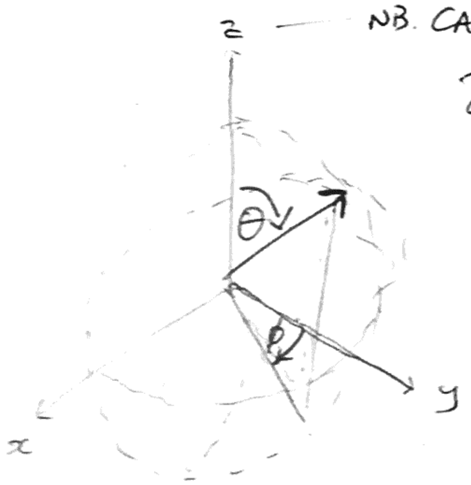
One possible (and useful) basis set is $\{|\uparrow\rangle, |\downarrow\rangle\}$ (also written as $\{|\alpha\rangle, |\beta\rangle\}$ or $\{|+\tfrac{1}{2}\rangle, |-\tfrac{1}{2}\rangle\}$)



$$|\Psi\rangle = c_1|\uparrow\rangle + c_2|\downarrow\rangle = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} a_1 + ib_1 \\ a_2 + ib_2 \end{pmatrix}$$

$|\Psi\rangle$ is described by 4 real numbers (degrees of freedom) – but how many are needed to describe an orientation?

NB. CARTESIAN COORDINATES!



2 : latitude and longitude!

What about the other 2 d.o.f.?

① Normalisation: $\langle\Psi|\Psi\rangle = 1$

$\Rightarrow c_1^* c_1 + c_2^* c_2 = 1$

② Phase: All possible observables are real numbers. Wavefunctions differing by a complex phase only are indistinguishable

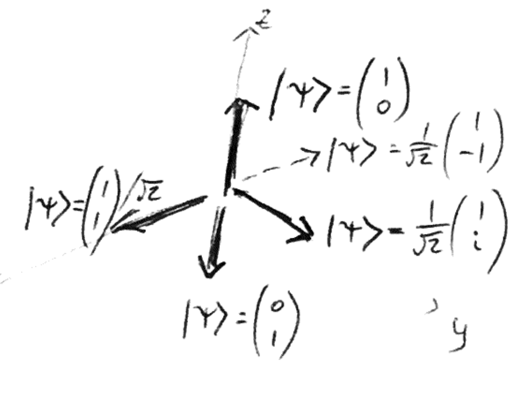
Rewrite $|\Psi\rangle = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} r_1 e^{i\theta_1} \\ r_2 e^{i\theta_2} \end{pmatrix}$

$r_1^2 + r_2^2 = 1$

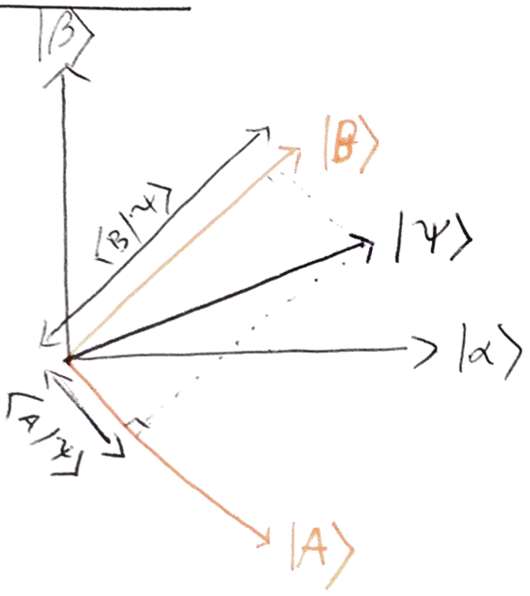
$\Rightarrow r_2 = \sqrt{1 - r_1^2}$

$$= \underbrace{e^{i\theta_1}}_{\text{unobservable}} \begin{pmatrix} r_1 \\ \sqrt{1-r_1^2}\, e^{i(\theta_2 - \theta_1)} \end{pmatrix}$$

$|\Psi\rangle$ only depends on relative amplitudes and phase <u>DIFFERENCE</u>.



EXPECTATIONS



$|\alpha\rangle, |\beta\rangle$ are eigenstates of $\hat{H} = \omega \hat{I}_z$

$|A\rangle, |B\rangle$ are eigenstates of $\hat{Q}$

Recap: What is outcome of measuring $\hat{Q}$ on state $|\Psi\rangle$?

Write $|\Psi\rangle$ in terms of $\hat{Q}$ eigenstates:

$$|\Psi\rangle = \Big(|A\rangle\langle A| + |B\rangle\langle B|\Big)|\Psi\rangle$$

$$= \underbrace{\langle A|\Psi\rangle\, |A\rangle}_{\text{get state } A \text{ and } \lambda_A \text{ with probability } |\langle A|\Psi\rangle|^2} + \underbrace{\langle B|\Psi\rangle\, |B\rangle}_{\text{get state } B \text{ and } \lambda_B \text{ with probability } |\langle B|\Psi\rangle|^2} = c_A|A\rangle + c_B|B\rangle$$

Result is always purely random – but what if we had many identical copies of $|\Psi\rangle$? Then we could calculate average or EXPECTATION:

$|A\rangle$ and $|B\rangle$ are eigenstates of $\hat{Q}$:

$$\hat{Q}|A\rangle = \lambda_A|A\rangle$$
$$\hat{Q}|B\rangle = \lambda_B|B\rangle$$

$$\langle \hat{Q} \rangle = \langle \Psi|\hat{Q}|\Psi\rangle$$

$$= \left(c_A^*\langle A| + c_B^*\langle B|\right)\hat{Q}\left(c_A|A\rangle + c_B|B\rangle\right)$$

$$= \left(c_A^*\langle A| + c_B^*\langle B|\right)\left(c_A\lambda_A|A\rangle + c_B\lambda_B|B\rangle\right)$$

$$= \underbrace{c_A^* c_A}_{\text{This is } P(A) \text{ for a single measurement!}} \cdot \lambda_A + \underbrace{c_B^* c_B}_{P(B)} \cdot \lambda_B$$

because $\langle A|A\rangle = 1$, $\langle A|B\rangle = 0$ etc (orthonormality)

Can also write in matrix notation (in $\{|A\rangle, |B\rangle\}$ basis)

$$\langle Q\rangle = \langle \Psi|\hat{Q}|\Psi\rangle = \begin{pmatrix} c_A^* & c_B^* \end{pmatrix} \underbrace{\begin{pmatrix} \cdot & ? & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}}_{\hat{Q} = \begin{pmatrix} \lambda_A & 0 \\ 0 & \lambda_B \end{pmatrix}} \begin{pmatrix} c_A \\ c_B \end{pmatrix}$$

Operators are DIAGONAL when written <u>in their own</u> eigenbasis.

What happens if we only know $|\Psi\rangle$ in the $\hat{\mathcal{H}}$ eigenbasis?
Nothing! Geometric picture is exactly the same! (Just more algebra!)

$$|\Psi\rangle = c_\alpha|\alpha\rangle + c_\beta|\beta\rangle = \big(|A\rangle\langle A| + |B\rangle\langle B|\big)\big(c_\alpha|\alpha\rangle + c_\beta|\beta\rangle\big)$$

$$= c_\alpha\langle A|\alpha\rangle|A\rangle + c_\beta\langle A|\beta\rangle|A\rangle + c_\alpha\langle B|\alpha\rangle|B\rangle + c_\beta\langle B|\beta\rangle|B\rangle$$

$$= c_A|A\rangle + c_B|B\rangle, \quad c_A = c_\alpha\langle A|\alpha\rangle + c_\beta\langle A|\beta\rangle$$
$$c_B = c_\alpha\langle B|\alpha\rangle + c_\beta\langle B|\beta\rangle$$

ie. rewrite in $\hat{Q}$ eigenbasis.

$$\langle\Psi| = c_A^*\langle A| + c_B^*\langle B|$$

$$\therefore \langle Q\rangle = \langle\Psi|Q|\Psi\rangle = \begin{pmatrix} c_A^* & c_B^* \end{pmatrix}\begin{pmatrix} \lambda_A & 0 \\ 0 & \lambda_B \end{pmatrix}\begin{pmatrix} c_A \\ c_B \end{pmatrix}$$

$$= c_A^* c_A \lambda_A + c_B^* c_B \lambda_B$$

$$= \big(c_\alpha^*\langle A|\alpha\rangle^* + c_\beta^*\langle A|\beta\rangle\big)\big(c_\alpha\langle A|\alpha\rangle + c_\beta\langle A|\beta\rangle\big)\lambda_A + \ldots\lambda_B$$

$$= c_\alpha^* c_\alpha\big(\langle A|\alpha\rangle^*\langle A|\alpha\rangle\lambda_A + \langle B|\alpha\rangle^*\langle B|\alpha\rangle\lambda_B\big)$$
$$+ c_\alpha^* c_\beta\big(\langle A|\alpha\rangle^*\langle B|\alpha\rangle\lambda_A + \langle B|\alpha\rangle^*\langle B|\beta\rangle\lambda_B\big)$$

$$= \begin{pmatrix} c_\alpha^* & c_\beta^* \end{pmatrix}\underbrace{\begin{pmatrix} \langle A|\alpha\rangle^*\langle A|\alpha\rangle\lambda_A + \langle B|\alpha\rangle^*\langle B|\alpha\rangle\lambda_B & \cdots \\ \cdots & \cdots \end{pmatrix}}_{\text{matrix representation of } \hat{Q} \text{ in } \hat{\mathcal{H}} \text{ eigenbasis.}}\begin{pmatrix} c_\alpha \\ c_\beta \end{pmatrix}$$

NB. not diagonal!

TIME EVOLUTION: THE SCHRÖDINGER EQUATION

Fundamental postulate: $i\hbar \frac{d}{dt}|\Psi\rangle = \hat{H}|\Psi\rangle$

Looks like an eigenvalue equation, but $|\Psi\rangle$ does not need to be eigenstate of $\hat{H}$.

Eigenvalues of $\hat{\mathcal{H}}$ do however form complete orthonormal basis set (because it's an observable operator) so we can write $|\Psi\rangle$ in terms of $\hat{\mathcal{H}}$ eigenstates $\{|n\rangle\}$: $\hat{\mathcal{H}}|n\rangle = E_n|n\rangle$

$$|\Psi\rangle = \sum_n c_n|n\rangle \qquad c_n = \langle n|\Psi\rangle \qquad \left(\text{recall: } \sum_n |n\rangle\langle n| = 1\right)$$

Suppose $|\Psi\rangle$ is an eigenstate of $\hat{\mathcal{H}}$, $|\Psi\rangle = |n\rangle$:

then $i\hbar \frac{d}{dt}|\Psi\rangle = i\hbar\frac{d}{dt}|n\rangle = \hat{\mathcal{H}}|n\rangle = E_n|n\rangle$

$$\Rightarrow \frac{d}{dt}|n\rangle = \frac{E_n}{i\hbar}|n\rangle = -\frac{i}{\hbar}E_n|n\rangle$$

All you need to know about differential equations:

General solution of $\frac{d}{dt}y(t) = Ay(t)$

is $y(t) = e^{At}y(0)$

so $|n(t)\rangle = \underbrace{e^{-iE_nt/\hbar}}_{\text{phase factor}}|n(0)\rangle$

i.e. eigenstates of $\hat{\mathcal{H}}$ just change complex phase at a rate proportional to their energy

If $|\Psi\rangle = |n\rangle$, this phase change is <u>undetectable</u>.

But what if $|\Psi\rangle$ is not an eigenstate of $\hat{\mathcal{H}}$?

Suppose $|\psi\rangle$ is a combination (superposition) of $\hat{H}$ eigenstates $|\alpha\rangle$ and $|\beta\rangle$

$$|\psi\rangle = c_\alpha |\alpha\rangle + c_\beta |\beta\rangle$$

$$i\hbar \frac{d}{dt}|\psi\rangle = i\hbar \frac{d}{dt}\left(c_\alpha|\alpha\rangle + c_\beta|\beta\rangle\right) = \hat{H}\left(c_\alpha|\alpha\rangle + c_\beta|\beta\rangle\right)$$

$$= c_\alpha E_\alpha |\alpha\rangle + c_\beta E_\beta |\beta\rangle$$

Multiply from left by $\langle\alpha|$ and use orthonormality:

$$i\hbar \frac{dc_\alpha}{dt} = c_\alpha E_\alpha \implies c_\alpha(t) = e^{-iE_\alpha/\hbar} c_\alpha(0)$$

and $\langle\beta|$: $$i\hbar \frac{dc_\beta}{dt} = c_\beta E_\beta \implies c_\beta(t) = e^{-iE_\beta/\hbar} c_\beta(0)$$

$$\therefore |\psi(t)\rangle = \begin{pmatrix} e^{-iE_\alpha t/\hbar} c_\alpha(0) \\ e^{-iE_\beta t/\hbar} c_\beta(0) \end{pmatrix}$$

i.e. each component picks up phase according to its energy.

Are these phase factors observable?

Yes! DIFFERENCES are observable!

$$|\psi(t)\rangle = e^{-iE_\alpha t/\hbar} \cdot \begin{pmatrix} c_\alpha(0) \\ e^{-i(E_\alpha - E_\beta)t/\hbar} c_\beta(0) \end{pmatrix}$$

Example: $\hat{H} = -\omega\hbar \hat{I}_z = \begin{pmatrix} -\omega\hbar/2 & 0 \\ 0 & +\omega\hbar/2 \end{pmatrix}$

$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (spin along x-axis)

$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{i\omega t} \end{pmatrix}$ (ignoring phase factor)

so $\underset{x}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \rightarrow \underset{y}{\begin{pmatrix} 1 \\ i \end{pmatrix}} \rightarrow \underset{-x}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} \rightarrow \underset{-y}{\begin{pmatrix} 1 \\ -i \end{pmatrix}} \rightarrow \underset{x}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

Larmor precession!

Other operators have representations in the Zeeman basis (another name for ~~Hamiltonian~~ basis $\{\langle\alpha|, \langle\beta|\}$):

$$I_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \underbrace{I_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad I_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}}_{\text{could use to calculate precession during pulses like earlier}}$$

also: $I_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad I_\beta = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad I_+ = I_x + iI_y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad I_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

PURE AND MIXED STATES

We now know how to write pure states,

e.g. $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad |\rightarrow\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

superposition of $|\uparrow\rangle$ and $|\downarrow\rangle$ leads to quantum uncertainty when measuring I_z

But how do we represent statistical uncertainty?

e.g. $|50\%\text{ chance of }\uparrow, 50\%\text{ chance of }\downarrow\rangle = ?$

$\frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right)$? but that is just $|\rightarrow\rangle$

This is a mixed state – cannot be written as any combination of basis vectors.

– can be represented using a DENSITY MATRIX

DENSITY MATRICES/OPERATORS

If we have a mixture $\{p_i, |\psi_i\rangle\}$ (not restricted to eigenstates) we can write the density operator:

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

For above example, 50% $|\uparrow\rangle$ and 50% $|\downarrow\rangle$:

$$\rho = \tfrac{1}{2}|\uparrow\rangle\langle\uparrow| + \tfrac{1}{2}|\downarrow\rangle\langle\downarrow|$$
$$= \tfrac{1}{2}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + \tfrac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}$$
$$= \tfrac{1}{2}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + \tfrac{1}{2}\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} = \tfrac{1}{2}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

off-diagonals don't need to be zero, eg: pure $|\rightarrow\rangle$: $\rho = \tfrac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1 & 1\end{pmatrix} = \tfrac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}$

diagonal ALWAYS adds up to 1: $\mathrm{Tr}\,\rho = 1$

~~Measurement: Probability of system in state $|x\rangle$:~~

~~$P(|x\rangle) = \sum_j p_j |\langle\psi_j|x\rangle|^2 = \sum_j p_j \langle x|\psi_j\rangle\langle\psi_j|x\rangle$~~

When $|\psi\rangle$ given in terms of basis $|\alpha\rangle, |\beta\rangle$:

$$\rho = \begin{pmatrix}\overline{c_\alpha^* c_\alpha} & \overline{c_\alpha^* c_\beta}\\ \overline{c_\beta^* c_\alpha} & \overline{c_\beta^* c_\beta}\end{pmatrix}$$ where $\overline{c_\alpha^* c_\alpha}$ indicates ensemble average

diagonals → "populations"
off-diagonals → "coherences"
but these labels only make sense when written in $\hat{H}$ eigenbasis.

Expectation values of density matrices:

$$\langle Q\rangle = \sum_j p_j \langle j|Q|j\rangle \quad \text{(definition of ensemble averaged expectation)}$$

Insert two complete basis sets:

$$\Rightarrow \langle Q\rangle = \sum_j p_j \sum_{\mu,\nu} \langle j|\mu\rangle\langle\mu|Q|\nu\rangle\langle\nu|j\rangle$$
$$= \sum_{\mu,\nu}\Big(\sum_j p_j \langle\nu|j\rangle\langle j|\mu\rangle\Big)\langle\mu|Q|\nu\rangle$$
$$= \sum_{\mu,\nu}\langle\nu|\rho\underbrace{|\mu\rangle\langle\mu|}_{=1}Q|\nu\rangle$$
$$= \sum_\nu \langle\nu|\rho Q|\nu\rangle = \mathrm{Tr}(\rho Q)$$

Time evolution of density matrix:

TDSE: $i\hbar \frac{\partial}{\partial t}|\psi\rangle = \hat{\mathcal{H}}|\psi\rangle$ and h.c. $-i\hbar\frac{\partial}{\partial t}\langle\psi| = \langle\psi|\hat{\mathcal{H}}$

$$i\hbar \frac{\partial}{\partial t}\rho = i\hbar\frac{\partial}{\partial t}\sum_j p_j |j\rangle\langle j| = i\hbar\sum_j p_j\left(\frac{\partial|j\rangle}{\partial t}\langle j| + |j\rangle\frac{\partial\langle j|}{\partial t}\right)$$

$$= i\hbar\sum_j p_j\left(-\frac{i}{\hbar}\hat{\mathcal{H}}|j\rangle\langle j| + \frac{i}{\hbar}|j\rangle\langle j|\hat{\mathcal{H}}\right)$$

$$= \hat{\mathcal{H}}\rho - \rho\hat{\mathcal{H}} = [\hat{\mathcal{H}}, \hat{\rho}]$$

Liouville-von Neumann equation

This is solved by $\hat{\rho}(t) = e^{-i\hat{\mathcal{H}}t}\hat{\rho}(0)e^{+i\hat{\mathcal{H}}t}$

Commutators and matrix exponentials

The commutator $[A,B] = AB - BA$

If $[A,B]=0$, A and B are said to <u>commute</u> and their order can be interchanged.

Nothing odd about this! Rotations do not commute.

Important to simplify calculations,
eg. if $[\hat{\mathcal{H}},\rho]=0$, $\frac{d\rho}{dt}=0$! No time evolution!

Commutation important when calculating matrix exponentials:

$$e^{A+B} \neq e^A e^B$$ <u>unless</u> A and B commute

eg. if Hamiltonian has two parts, $\hat{\mathcal{H}} = \hat{H}_A + \hat{H}_B$, can only calculate time evolution for A and B separately if H_A and H_B commute:

$$\hat{\rho}(t) = e^{-i(H_A+H_B)t}\hat{\rho}(0)e^{+i(H_A+H_B)t} \neq e^{-iH_At}e^{-iH_Bt}\rho(0)e^{+iH_At}e^{+iH_Bt}$$

(in general)

Recap: Matrix representation

All operators can be written in a matrix representation (for a specific basis set) such that

$$\langle \hat{Q} \rangle = \langle \Psi | \hat{Q} | \Psi \rangle = \begin{pmatrix} \langle\alpha| & \langle\beta| \end{pmatrix} \cdot \hat{Q} \cdot \begin{pmatrix} |\alpha\rangle \\ |\beta\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \langle\alpha|Q|\alpha\rangle & \langle\beta|Q|\alpha\rangle \\ \langle\alpha|Q|\beta\rangle & \langle\beta|Q|\beta\rangle \end{pmatrix} = \begin{pmatrix} c_\alpha^* c_\alpha & c_\beta^* c_\alpha \\ c_\alpha^* c_\beta & c_\beta^* c_\beta \end{pmatrix}$$

Important examples:

$$\hat{E} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad I_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad I_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad I_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

One-pulse experiment:

Density matrix at equilibrium: $\hat{\rho}_0 = \dfrac{e^{-\hat{\mathcal{H}}/kT}}{Z}$ ← normalisation

$$\hat{\rho}_0 = \frac{1}{Z}\begin{pmatrix} \langle\alpha|e^{-\hat{\mathcal{H}}/kT}|\alpha\rangle & \langle\alpha|e^{-\mathcal{H}/kT}|\beta\rangle \\ \langle\beta|e^{-\mathcal{H}/kT}|\alpha\rangle & \langle\beta|e^{-\mathcal{H}/kT}|\beta\rangle \end{pmatrix}$$

$e^{-\hat{\mathcal{H}}/kT}|\alpha\rangle = e^{-E_\alpha/kT}|\alpha\rangle$ etc.

$$= \frac{1}{Z}\begin{pmatrix} e^{-E_\alpha/kT} & 0 \\ 0 & e^{-E_\beta/kT} \end{pmatrix}$$

$e^{\alpha} \approx 1 + \alpha + O(\alpha^2)$

$$= \frac{1}{Z}\begin{pmatrix} 1+\frac{1}{2}\hbar\omega_0/kT & 0 \\ 0 & 1-\frac{1}{2}\hbar\omega_0/kT \end{pmatrix}$$

$E_\alpha = -\frac{1}{2}\hbar\omega_0$
$E_\beta = +\frac{1}{2}\hbar\omega_0$

$$= \frac{1}{Z}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{Z}\cdot\frac{\hbar\omega_0}{2kT}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{1}{Z}\left(E + \frac{\hbar\omega}{2kT} I_z\right) \longrightarrow I_z$$

ignoring constants and identity E

Now consider rotation about x-axis (x-pulse)

$$\mathcal{H}_1 = \omega_1 I_x = \frac{\omega_1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\frac{\pi}{2} \text{ pulse} \Rightarrow \tau_p = \frac{\pi}{2}\frac{1}{\omega_1}$$

$$e^{-i\mathcal{H}_1 t_p} = e^{-i\frac{\pi}{2} I_x} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \qquad e^{+i\mathcal{H}_1 t_p} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

$$\rho_0 \longrightarrow \rho_1 = e^{-i\mathcal{H}_1 t_p}\rho_0 e^{i\mathcal{H}_1 t_p}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \cdot \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = -I_y$$

$$\langle I_y \rangle = \mathrm{Tr}(\rho I_y) = \mathrm{Tr}(-I_y \cdot I_y) = \mathrm{Tr}\left[-\frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \cdot \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\right]$$

$$= -1/2$$

<u>Product operator expansion:</u>

$$\hat{\rho} = a_1 \hat{I}_x + a_2 \hat{I}_y + a_3 \hat{I}_z$$

write $\hat{\rho}$ in terms of 'component' matrices, can simplify calculations:

$$\hat{\rho} \longrightarrow e^{-i\mathcal{H}t}\hat{\rho}\, e^{+i\mathcal{H}t} = e^{-i\mathcal{H}t}(\rho_1 + \rho_2 + \dots)e^{+i\mathcal{H}t}$$

$$= e^{-i\mathcal{H}t}\rho_1 e^{+i\mathcal{H}t} + e^{-i\mathcal{H}t}\rho_2 e^{+i\mathcal{H}t} + \dots \text{ etc}$$

can calculate evolution of components separately.